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Thermoelastic field of a transversely isotropic elastic medium containing a penny-shaped crack: exact fundamental solution

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Abstract

The three-dimensional steady-state basic equations of thermoelasticity for a transversely isotropic elastic medium are simplified by introducing two displacement functions. Using the operator theory, a static general solution is obtained, which is expressed in terms of four quasi-harmonic functions. Potential theory method is extended to account for the thermal effect for crack problems. Exact and complete fundamental solution is derived for the problem of a penny-shaped crack subjected to a point-temperature load, arbitrarily acting on the crack surface. This is completely new to the literature. In the case of uniform-temperature load, exact expressions are also obtained for the three-dimensional thermoelastic field. Comparison with the existent results shows a good agreement. All expressions of the thermoelastic field in the full space are obtained in terms of elementary functions, which can facilitate their further usage.

Keywords: Transversely isotropic; Thermoelastic general solution; Potential theory method; Penny-shaped crack

1. Introduction

The study of thermoelastic problems has always been an important branch in solid mechanics (Nowacki, 1962; Nowinski, 1978). In particular, the thermoelastic fracture problems subjected to various types of thermal boundary conditions have been discussed extensively in the literature (Sih, 1962; Wilson and Yu, 1979; Prasad et al., 1994; Georgiadis et al., 1998; Kotousov, 2002). As regards the steady-state problem of penny-shaped crack in isotropic elastic media, most analytical works treated the axisymmetric case, for which Hankel transform technique and the theory of dual integral equations were usually employed (Sneddon and Lowengrub, 1969; Kassir and Sih, 1975). Shail (1964) proposed a different solution method by virtue of a general solution with zero shear stress on the crack plane. The general solution is expressed by two harmonic functions, one of which is directly related to the temperature field. No report on

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a penny-shaped crack subjected to point-temperature load arbitrarily located at the crack surface could be found yet in literature, although the corresponding fundamental solution can play an important role in boundary element method (BEM) thermoelastic facture analysis.

Some metallic materials, such as zinc and magnesium, are transversely isotropic (Hermon, 1961). Many fibrous composites may be also modeled as transversely isotropic materials (Christensen, 1979). There have been many reports on three-dimensional analysis of transversely isotropic thermoelastic materials. Sharma (1958) extended the method of Elliott (1948) to thermoelasticity and determined the stresses due to temperature in a semi-infinite transversely isotropic solid. Singh (1960) solved a number of problems on axisymmetric thermal stresses in a semi-space utilizing two displacement functions. Employing the Hankel transform method, Mehta (1966) investigated the thermal stress around a crack in an elastic solid of transversely isotropic material, but most final expressions were given in integral form. Tsai (1983a,b) used a similar method to study the thermal stress in a transversely isotropic medium containing a penny-shaped crack; the expressions for normal stress and axial displacement at the plane of crack surface were explicitly derived. Singh et al. (1987) examined the steady-state thermoelastic behavior of an external circular crack subjected to a symmetric temperature distribution at the crack surfaces. Podil'chuk and Sokolovskii (1994) employed a general solution for transversely isotropic elasticity to solve the steady-state problem of an infinite medium with an internal elliptical crack by the trial-and-error method. Later, Podil'chuk and Dobrivecher (1996a,b) considered the inclusion problem using a similar process. Noda and Ashida (1987, 1994) investigated the dynamic problems of a penny-shaped crack in a transversely isotropic infinite solid or in a transversely isotropic cylinder. Recently, Tsai (1998, 2000) extended his earlier works (Tsai, 1983a,b) to the flat toroidal crack case using the techniques of triple integral equations and multiplying factors.

It is noted that for transversely isotropic materials without thermal effect, Fabrikant (1989, 1991) has successfully developed a so-called potential theory method which can be widely used in three-dimensional analysis of crack and punch problems. The method is based on the general solution proposed by Elliott (1948) that contains three quasi-harmonic potential functions. Exact solutions can be obtained for some non-classical problems using this method. For example, complete expressions have been derived for the elastic field in the full space for the problem of a transversely isotropic elastic body containing a penny-shaped crack, which is subjected to normal or shear point load at its crack surface. Chen and Shioya (1999, 2000) and Chen (2000) extended the Fabrikant's potential theory method to piezoelasticity and obtained some exact solutions of crack and punch problems for piezoelectric materials. However, there is no parallel work in the domain of thermoelasticity. This may be mainly due to the lack of a thermoelastic general solution expressed by quasi-harmonic functions as that for purely elastic materials (Elliott, 1948).

The general solution for thermoelasticity employed by Podil'chuk and Sokolovskii (1994) involves four potential functions, three of them are quasi-harmonic and the remainder one satisfies a differential equation with inhomogeneous term. Ding et al. (1997) also presented a general solution of transversely isotropic thermoelasticity, which consisted of two parts, i.e. the particular solution and the general solution. The particular solution can be solved from the heat conduction equation and the corresponding boundary conditions. The general part is identical to those of the purely elastic one. Either the general solution employed by Podil'chuk and Sokolovskii (1994) or the one proposed by Ding et al. (1997) is in fact a simple extension of that for the purely elastic one (Elliott, 1948) that the temperature field should be solved independently and a priori. Note that for transient problems, Ashida et al. (1993) proposed a general solution technique for which the temperature field also should be solved in advance. Employing such general solutions, however, we can not use many splendid results obtained by Fabrikant for transversely isotropic elasticity (Fabrikant, 1989, 1991).

In this paper, two displacement functions are introduced to simplify the basic three-dimensional equations of thermoelasticity with transverse isotropy for the steady-state problem. Using the operator theory, we derive a general solution that is expressed in terms of two functions: One satisfies a quasi-harmonic equation and the other satisfies a six-order partial differential equation. By virtue of the generalized Almansi's theorem, the general solution is further expressed in terms of four quasi-harmonic functions. The steady-state thermal stresses of a transversely isotropic material containing a penny-shaped crack are then investigated. The problems can be turned to solve the corresponding mixed boundary-value problems of a half space and thus the potential theory method proposed by Fabrikant (1989) is generalized to account for the thermal effect. An integral equation and an integro-differential equation are then derived. When the crack is subjected to point-temperature load or uniform-temperature load at its crack surface, complete and exact expressions of the three-dimensional thermoelastic field are obtained in terms of elementary functions. For the uniform-temperature load case, comparison is made with the results obtained by Tsai (1983a) and good agreement is obtained. It is noted that most expressions derived in the paper have not been reported before in literature. Especially, the solution corresponding to the point-temperature load is entirely new and can be used as fundamental solution in BEM. It also can be used to construct analytical solution for an arbitrary temperature load at the crack surface. The stress intensity factor for a penny-shaped crack subjected to an arbitrary temperature load is given in the paper as an example.

2. Basic equations

In Cartesian coordinates (x, y, z), the Duhamel–Neumann relations for a transversely isotropic elastic medium with the isotropic plane parallel with the plane x–y are (Tsai, 1983a,b)

$$\begin{aligned}
\sigma_x &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} - \beta_1 T,\\
\sigma_y &= c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} - \beta_1 T,\\
\sigma_z &= c_{13}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + c_{33}\frac{\partial w}{\partial z} - \beta_3 T,\\
\tau_{yz} &= c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \tau_{zx} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \tau_{xy} = \frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned} \tag{1}$$

where $u(v, w)$ and $\sigma_i(\tau_{ij})$ are components of displacement and stress, respectively; T is the temperature change with $T = 0$ corresponding to the free-stress state; c_{ij} and β_i are elastic constants and thermal moduli, respectively.

The temperature field in the medium in a steady-state is governed by the following equation:

$$\left(k_{11}\varDelta + k_{33}\frac{\partial^2}{\partial z^2}\right)T = 0, \tag{2}$$

where $\varDelta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the planar Laplacian, and k_{ij} are the coefficients of thermal conductivity.

The equilibrium equations in terms of displacement are shown to be

$$\begin{aligned}
&\left(c_{11}\frac{\partial^2}{\partial x^2} + c_{66}\frac{\partial^2}{\partial y^2} + c_{44}\frac{\partial^2}{\partial z^2}\right)u + (c_{12} + c_{66})\frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial x \partial z} - \beta_1\frac{\partial T}{\partial x} = 0,\\
&(c_{12} + c_{66})\frac{\partial^2 u}{\partial x \partial y} + \left(c_{66}\frac{\partial^2}{\partial x^2} + c_{11}\frac{\partial^2}{\partial y^2} + c_{44}\frac{\partial^2}{\partial z^2}\right)v + (c_{13} + c_{44})\frac{\partial^2 w}{\partial y \partial z} - \beta_1\frac{\partial T}{\partial y} = 0,\\
&(c_{13} + c_{44})\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(c_{44}\varDelta + c_{33}\frac{\partial^2}{\partial z^2}\right)w - \beta_3\frac{\partial T}{\partial z} = 0,
\end{aligned} \tag{3}$$

where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$.

3. Static general solution

Two displacement functions ψ and G are introduced as follows

$$u = \frac{\partial \psi}{\partial y} - \frac{\partial G}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x} - \frac{\partial G}{\partial y}. \tag{4}$$

It is then obtained from Eqs. (3) and (2) that

$$\left(c_{66}\Delta + c_{44}\frac{\partial^2}{\partial z^2}\right)\psi = 0, \tag{5}$$

$$\mathbf{D}\begin{Bmatrix} G \\ w \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{6}$$

where $\mathbf{D}$ is the following operator matrix:

$$\mathbf{D} = \begin{bmatrix} c_{11}\Delta + c_{44}\frac{\partial^2}{\partial z^2} & -(c_{13}+c_{44})\frac{\partial}{\partial z} & \beta_1 \\ -(c_{13}+c_{44})\Delta\frac{\partial}{\partial z} & c_{44}\Delta + c_{33}\frac{\partial^2}{\partial z^2} & -\beta_3\frac{\partial}{\partial z} \\ 0 & 0 & k_{11}\Delta + k_{33}\frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{7}$$

and we have

$$|\mathbf{D}| = \left(a_0\frac{\partial^4}{\partial z^4} + b_0\Delta\frac{\partial^2}{\partial z^2} + c_0\Delta^2\right)\left(k_{11}\Delta + k_{33}\frac{\partial^2}{\partial z^2}\right), \tag{8}$$

where

$$a_0 = c_{33}c_{44}, \quad b_0 = c_{11}c_{33} + c_{44}^2 - (c_{13}+c_{44})^2, \quad c_0 = c_{11}c_{44}.$$

The derivation presented above is similar to that described in Ding et al. (1997); the difference is that the equation of heat conduction and the two equations governing G and w are combined together as shown in Eq. (6). It should be noted that in almost all previous works (Ashida et al., 1993; Podil'chuk and Sokolovskii, 1994; Ding et al., 1997), the temperature T should be solved independently from the thermal conduction equation, i.e. Eq. (2), which usually leads to general solutions that include a particular part associated with the temperature.

By virtue of the operator theory, we obtain the following general solutions

$$G = A_{i1}F, \quad w = A_{i2}F, \quad T = A_{i3}F, \quad (i = 1, 2, 3), \tag{9}$$

where A_{ij} are the algebraic cominors of $\mathbf{D}$, and the function F satisfies

$$|\mathbf{D}|F = \left(\Delta + \frac{\partial^2}{\partial z_1^2}\right)\left(\Delta + \frac{\partial^2}{\partial z_2^2}\right)\left(\Delta + \frac{\partial^2}{\partial z_3^2}\right)F = 0, \tag{10}$$

where $z_i = s_i z$, $s_3 = \sqrt{k_{11}/k_{33}}$, and s_1 and s_2 are the roots with positive real part of the following eigen-equation

$$a_0 s^4 - b_0 s^2 + c_0 = 0. \tag{11}$$

It can be seen that, if we take $i = 1$ and 2 in Eq. (9), then we will get two general solutions both implying $T = 0$, which are actually identical to the ones without thermal effect (Elliott, 1948; Ding et al., 1997). Taking $i = 3$ and writing out the expressions for A_{3j}, we obtain

$$u = \frac{\partial\psi}{\partial y} - \left(a_1\Delta + b_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial x}, \quad v = -\frac{\partial\psi}{\partial x} - \left(a_1\Delta + b_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial y},$$
$$w = \left(a_2\Delta + b_2\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial z}, \quad T = \left(a_0\frac{\partial^4}{\partial z^4} + b_0\Delta\frac{\partial^2}{\partial z^2} + c_0\Delta^2\right)F, \tag{12}$$

where

$$a_1 = -\beta_1 c_{44}, \quad b_1 = \beta_3(c_{13} + c_{44}) - \beta_1 c_{33}, \quad a_2 = \beta_3 c_{11} - \beta_1(c_{13} + c_{44}), \quad b_2 = \beta_3 c_{44}.$$

In cylindrical coordinates (r, ϕ, z), the general solution can be easily obtained. In fact, the expressions for w and T are identical to that in Eq. (12), while those for the radial and circumferential displacements u_r and u_ϕ are, respectively

$$u_r = \frac{\partial\psi}{r\partial\phi} - \left(a_1\Delta + b_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial r}, \quad u_\phi = -\frac{\partial\psi}{\partial r} - \left(a_1\Delta + b_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{r\partial\phi}, \tag{13}$$

here $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\phi^2$ is the Laplacian in polar coordinates.

Using the generalized Almansi's theorem (Ding et al., 1996), the function F can be expressed in terms of three quasi-harmonic equations

$$F = \begin{cases} F_1 + F_2 + F_3, & \text{for distinct } s_i, \\ F_1 + F_2 + zF_3, & \text{for } s_1 \neq s_2 = s_3, \\ F_1 + zF_2 + z^2F_3, & \text{for } s_1 = s_2 = s_3, \end{cases} \tag{14}$$

where F_i satisfy, respectively,

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2}\right)F_i = 0, \quad (i = 1, 2, 3). \tag{15}$$

It is noted that only in Eq. (14) of this paper, we have not specified s_3 to the value of $\sqrt{k_{11}/k_{33}}$. For the sake of simplicity, we proceed to consider the case of distinct eigenvalues here and after. In this case, the general solution has the simplest form of

$$u = \frac{\partial\psi}{\partial y} - \sum_{i=1}^{3}\lambda_{i1}\frac{\partial^3 F_i}{\partial x\partial z_i^2}, \quad v = -\frac{\partial\psi}{\partial x} - \sum_{i=1}^{3}\lambda_{i1}\frac{\partial^3 F_i}{\partial y\partial z_i^2},$$
$$w = \sum_{i=1}^{3}\lambda_{i2}s_i\frac{\partial^3 F_i}{\partial z_i^3}, \quad T = \lambda_{33}\frac{\partial^4 F_3}{\partial z_3^4}, \tag{16}$$

where $\lambda_{ij} = -a_j + b_j s_i^2$ $(j = 1, 2)$, $\lambda_{33} = a_0 s_3^4 - b_0 s_3^2 + c_0$. It is now assumed that

$$\lambda_{i1}\frac{\partial^2 F_i}{\partial z_i^2} = \psi_i, \quad (i = 1, 2, 3), \tag{17}$$

and writing $\psi_0 = \psi$, then Eq. (16) can be further simplified to

$$u = \frac{\partial\psi_0}{\partial y} - \sum_{i=1}^{3}\frac{\partial\psi_i}{\partial x}, \quad v = -\frac{\partial\psi_0}{\partial x} - \sum_{i=1}^{3}\frac{\partial\psi_i}{\partial y},$$
$$w = \sum_{i=1}^{3}\alpha_{i1}\frac{\partial\psi_i}{\partial z_i}, \quad T = \sum_{i=1}^{3}\alpha_{i2}\frac{\partial^2\psi_i}{\partial z_i^2}, \tag{18}$$

where $\alpha_{i1} = \lambda_{i2}s_i/\lambda_{i1}$, $\alpha_{12} = \alpha_{22} = 0$, $\alpha_{32} = \lambda_{33}/\lambda_{31}$, and

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2}\right)\psi_i = 0, \quad (i = 0, 1, 2, 3), \tag{19}$$

in which $z_0 = s_0 z$, and $s_0 = \sqrt{c_{66}/c_{44}}$.

In cylindrical coordinates (r, ϕ, z), the expressions for w and T are still given as in Eq. (18), while those for the radial and circumferential displacements are, respectively

$$u_r = \frac{\partial\psi_0}{r\partial\phi} - \sum_{i=1}^{3}\frac{\partial\psi_i}{\partial r}, \quad u_\phi = -\frac{\partial\psi_0}{\partial r} - \sum_{i=1}^{3}\frac{\partial\psi_i}{r\partial\phi}. \tag{20}$$

In order to generalize the potential theory method proposed by Fabrikant (1989) to the case of thermoelasticity, we introduce the following complex quantities

$$U = u + \mathrm{i}v, \quad \sigma_1 = \sigma_x + \sigma_y, \quad \sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy}, \quad \tau_z = \tau_{xz} + \mathrm{i}\tau_{yz}.$$

We then obtain that

$$\begin{aligned} &U = -\Lambda\left(\sum_{i=1}^{3}\psi_i + \mathrm{i}\psi_0\right), \quad \sigma_z = \sum_{i=1}^{3}\gamma_{i1}\frac{\partial^2\psi_i}{\partial z_i^2}, \\ &\sigma_1 = \sum_{i=1}^{3}\gamma_{i2}\frac{\partial^2\psi_i}{\partial z_i^2}, \quad \sigma_2 = -2c_{66}\Lambda^2\left(\sum_{i=1}^{3}\psi_i + \mathrm{i}\psi_0\right), \\ &\tau_z = \Lambda\left(\sum_{i=1}^{3}\gamma_{i3}\frac{\partial\psi_i}{\partial z_i} - \mathrm{i}s_0 c_{44}\frac{\partial\psi_0}{\partial z_0}\right), \end{aligned} \tag{21}$$

where $\Lambda = \partial/\partial x + \mathrm{i}\partial/\partial x$, and

$$\gamma_{i1} = c_{13} + c_{33}s_i\alpha_{i1} - \beta_3\alpha_{i2}, \quad \gamma_{2i} = 2[(c_{11} - c_{66}) + c_{13}s_i\alpha_{i1} - \beta_1\alpha_{i2}], \quad \gamma_{i3} = -c_{44}s_i + c_{44}\alpha_{i1}.$$

It can be verified that $\gamma_{i1}s_i = \gamma_{i3}$.

4. Generalized potential theory method for thermoelastic crack problem

Consider an infinite transversely isotropic elastic body containing a penny-shaped crack of radius a. The crack is located in the plane $z = 0$, which is parallel with the isotropic plane. The cylindrical coordinate system (r, ϕ, z) is adopted with the origin at the center of the crack. It is assumed that the crack is subjected to an arbitrarily distributed temperature $\Theta(r, \phi)$ at the crack surface. Using the symmetric condition, the problem can be turned to a mixed boundary-value problem of a half space $z \geqslant 0$ with the following conditions at the surface $z = 0$ (Sneddon and Lowengrub, 1969; Tsai, 1983a):

$$\begin{aligned} &0 \leqslant r \leqslant a: \quad \sigma_z = 0, \ T = \Theta(r, \phi), \\ &a < r < \infty: \quad w = \partial T/\partial z = 0, \\ &0 \leqslant r < \infty: \quad \tau_z = 0. \end{aligned} \tag{22}$$

Extending the potential theory method (Fabrikant, 1989) to thermoelasticity, it is assumed that

$$\psi_0 = 0, \quad \psi_i(z) = h_{i1}H_1(z_i) + h_{i2}H_2(z_i), \quad (i = 1, 2, 3), \tag{23}$$

where h_{i1} and h_{i2} are undetermined constants, and

$$H_1(r, \phi, z) = \iint_S \frac{\omega(N)}{R(M, N)}\,\mathrm{d}S, \quad H_2(r, \phi, z) = \iint_S \vartheta(N)\{z\ln[R(M, N) + z] - R(M, N)\}\,\mathrm{d}S, \tag{24}$$

where S is the crack domain $0 \leqslant r \leqslant a$, ω and ϑ are the crack surface displacement $w(r,\phi,0)$ and temperature gradient $\partial T(r,\phi,z)/\partial z|_{z=0}$, respectively, $R(M,N)$ is the distance between the two points $M(r,\phi,z)$ and $N(\rho,\theta,0)$, and $N \in S$. As compared with the potential theory method for pure elasticity, a new potential H_2 has been introduced here to account for the thermal effect. To satisfy the zero-shear stress condition at $z=0$, we take

$$\sum_{i=1}^{3} \gamma_{i3} h_{ij} = 0, \quad (j = 1,2). \tag{25}$$

The following two equations hold on account of the property of the potential of a simple layer

$$\begin{aligned} r > a: &\quad \left.\frac{\partial H_1}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial^3 H_2}{\partial z^3}\right|_{z=0} = 0, \\ r < a: &\quad \left.\frac{\partial H_1}{\partial z}\right|_{z=0} = -2\pi\omega = -2\pi w(r,\phi,0), \quad \left.\frac{\partial^3 H_2}{\partial z^3}\right|_{z=0} = -2\pi\vartheta = -2\pi \left.\frac{\partial T(r,\phi,z)}{\partial z}\right|_{z=0}. \end{aligned} \tag{26}$$

Then from Eqs. (18), (23) and (26) as well as the second condition in Eq. (22), we obtain

$$\sum_{i=1}^{3} \alpha_{i1} h_{i1} = -\frac{1}{2\pi}, \quad \sum_{i=1}^{3} \alpha_{i1} h_{i2} = 0, \quad \sum_{i=1}^{3} \alpha_{i2} h_{i1} s_i = 0, \quad \sum_{i=1}^{3} \alpha_{i2} h_{i2} s_i = -\frac{1}{2\pi}. \tag{27}$$

Thus

$$\begin{Bmatrix} h_{1j} \\ h_{2j} \\ h_{3j} \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{13} & \gamma_{23} & \gamma_{33} \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12}s_1 & \alpha_{22}s_2 & \alpha_{32}s_3 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \delta_{1j} \\ \delta_{2j} \end{Bmatrix}, \quad (j = 1,2), \tag{28}$$

where δ_{ij} is the Kronecker delta. To meet the first condition in Eq. (22) demands

$$g_{11} \Delta \iint_S \frac{\omega(N)}{R(N_0,N)} \mathrm{d}S - g_{12} \iint_S \frac{\vartheta(N)}{R(N_0,N)} \mathrm{d}S = 0, \tag{29}$$

$$\iint_S \frac{\vartheta(N)}{R(N_0,N)} \mathrm{d}S = -2\pi s_3 \Theta(N_0), \tag{30}$$

where $g_{1j} = \sum_{i=1}^{3} \gamma_{i1} h_{ij}$, $R(N_0,N)$ is the distance between the two points $N_0(r_0,\phi_0,0)$ and $N(\rho,\theta,0)$, and both N_0, $N \in S$. By virtue of Eq. (30), Eq. (29) can be rewritten as

$$\Delta \iint_S \frac{\omega(N)}{R(N_0,N)} \mathrm{d}S = -2\pi s_3 g_{12} \Theta(N_0)/g_{11}. \tag{31}$$

It can be seen that Eq. (31) is similar to the governing equation for crack problem in elasticity while Eq. (30) is similar to the one for punch problem in elasticity (Fabrikant, 1989).

The solutions to Eqs. (31) and (30) can be obtained by directly using Fabrikant's results (Fabrikant, 1989)

$$\omega(r,\phi) = \frac{s_3 g_{12}}{\pi^2 g_{11}} \int_0^{2\pi} \int_0^a \frac{1}{R} \tan^{-1}\left(\frac{\eta}{R}\right) \Theta(r_0,\phi_0) r_0 \, \mathrm{d}r_0 \, \mathrm{d}\phi_0, \tag{32}$$

$$\vartheta(r,\phi) = \frac{2s_3}{\pi r} L(r) \frac{\mathrm{d}}{\mathrm{d}r} \int_r^a \frac{x \, \mathrm{d}x}{(x^2 - r^2)^{1/2}} L\left(\frac{1}{x^2}\right) \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{r_0 \, \mathrm{d}r_0}{(x^2 - r_0^2)^{1/2}} L(r_0) \Theta(r_0,\phi), \tag{33}$$

where

$$R = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)}, \quad \eta = \sqrt{a^2 - r^2}\sqrt{a^2 - r_0^2}\Big/a$$

and $L(\cdot)$ is an operator defined as follows:

$$L(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-k^2)f(\phi_0)}{1+k^2-2k\cos(\phi-\phi_0)}\,\mathrm{d}\phi_0. \tag{34}$$

Substituting Eq. (32) into the first equation in Eq. (24) gives

$$H_1(r,\phi,z) = \frac{s_3 g_{12}}{\pi^2 g_{11}}\int_0^{2\pi}\int_0^a K(r,\phi,z;r_0,\phi_0)\Theta(r_0,\phi_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{35}$$

where the so-called Green's function $K(r,\phi,z;r_0,\phi_0)$ has been obtained by Fabrikant (1989) as follows:

$$K(M;N_0) = K(r,\phi,z;r_0,\phi_0) = \int_0^{2\pi}\int_0^a \frac{1}{R(N,N_0)}\tan^{-1}\left[\frac{\sqrt{a^2-\rho^2}\sqrt{a^2-r_0^2}}{aR(N,N_0)}\right]\frac{\rho\,\mathrm{d}\rho\,\mathrm{d}\theta}{R(M,N)}. \tag{36}$$

Here $R(\cdot,\cdot)$ denotes the distance between respective points: $M(r,\phi,z)$, $N(\rho,\theta,0)$ and $N_0(r_0,\phi_0,0)$. The derivative of H_1 with respect to z can be obtained from Eq. (35),

$$\frac{\partial H_1}{\partial z} = -\frac{2s_3 g_{12}}{g_{11}\pi}\int_0^{2\pi}\int_0^a \frac{1}{R(M,N_0)}\tan^{-1}\left[\frac{h}{R(M,N_0)}\right]\Theta(r_0,\phi_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{37}$$

where $h = \sqrt{a^2-l_1^2}\sqrt{a^2-r_0^2}/a$ and $l_1 = \frac{1}{2}\left[\sqrt{(r+a)^2+z^2} - \sqrt{(r-a)^2+z^2}\right]$. It is mentioned here that various derivatives of the Green's function K can also be found in Fabrikant (1989) and are listed in Appendix A for the reader's convenience.

The expression for H_2 could not be obtained by directly employing Fabrikant's results. However, we can obtain

$$\frac{\partial H_2}{\partial z} = \iint_S \vartheta(N)\ln[R(M,N)+z]\,\mathrm{d}S = -\frac{2s_3}{\pi}\int_0^{2\pi}\int_0^a \Pi(r,\phi,z;r_0,\phi_0)\Theta(r_0,\phi_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{38}$$

where the Green's function $\Pi(r,\phi,z;r_0,\phi_0)$ has been obtained by Fabrikant (1989) for elastic contact problems as follows:

$$\begin{aligned}\Pi(r,\phi,z;r_0,\phi_0) = &-\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{\sqrt{a^2-r_0^2}}\left[\ln\left(\frac{a+\sqrt{a^2-l_1^2}}{l_1}\right) + \frac{1}{\sqrt{\varsigma-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\varsigma-1}}\right)\right.\\ &\left. + \frac{1}{\sqrt{\bar{\varsigma}-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar{\varsigma}-1}}\right)\right],\end{aligned} \tag{39}$$

where $\varsigma = r\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}/r_0$ and $R_0 = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi-\phi_0) + z^2}$.

5. Exact thermoelastic fundamental solution

In the following, we divide the thermoelastic field into two parts. The first part corresponds to the potential H_1, and the second part corresponds to another potential H_2, as shown in Eq. (23). For a penny-shaped crack subjected to a point-temperature load $T = \Theta_0$ at an arbitrary point $(r_0,\phi_0,0)$, the thermal load at the right-hand side of Eq. (31) can be seen as a generalized point mechanical force. Thus the expressions of the thermoelastic field of the first part can be obtained by directly employing the results in Fabrikant (1989), except for the coefficients, as follows:

$$
\begin{aligned}
U^{(1)} &= -\frac{2s_3g_{12}}{\pi g_{11}}\sum_{i=1}^{3} h_{i1}f_1(z_i)\Theta_0, \quad w^{(1)} = -\frac{2s_3g_{12}}{\pi g_{11}}\sum_{i=1}^{3}\alpha_{i1}h_{i1}f_2(z_i)\Theta_0,\\
\sigma_z^{(1)} &= \frac{2s_3g_{12}}{\pi g_{11}}\sum_{i=1}^{3}\gamma_{i1}h_{i1}f_3(z_i)\Theta_0, \quad \sigma_1^{(1)} = \frac{2s_3g_{12}}{\pi g_{11}}\sum_{i=1}^{3}\gamma_{i2}h_{i1}f_3(z_i)\Theta_0,\\
\sigma_2^{(1)} &= -\frac{4s_3g_{12}c_{66}}{\pi g_{11}}\sum_{i=1}^{3}h_{i1}f_4(z_i)\Theta_0, \quad \tau_z^{(1)} = \frac{2s_3g_{12}}{\pi g_{11}}\sum_{i=1}^{3}\gamma_{i3}h_{i1}f_5(z_i)\Theta_0,\\
T^{(1)} &= \frac{2s_3g_{12}}{g_{11}\pi}\sum_{i=1}^{3}\alpha_{i2}h_{i1}f_3(z_i)\Theta_0,
\end{aligned} \tag{40}
$$

where $f_i(z)$ are given in Appendix A.

We can not write out immediately the expressions of the thermoelastic field of the second part by directly citing the results in Fabrikant (1989). From Eq. (38), the expression for H_2 can be written as

$$
H_2 = -\frac{2s_3}{\pi}\int_0^{2\pi}\int_0^{a}\Psi(r,\phi,z;r_0,\phi_0)\Theta(r_0,\phi_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{41}
$$

where $\Psi(r,\phi,z;r_0,\phi_0) = \int \Pi(r,\phi,z;r_0,\phi_0)\,\mathrm{d}z$. The integration of Π with respect to z is very difficult to execute. However, the derivatives of the Green's function $\Pi(r,\phi,z;r_0,\phi_0)$ have been derived by Fabrikant (1989) and are listed in Appendix A, from which we can derive the derivates of Ψ as follows:

$$
\begin{aligned}
\Lambda\Psi &= g_1(z) = \frac{1}{t}\left\{\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{z}{h} + \frac{1}{\sqrt{a^2-r_0^2}}\left[\sqrt{a^2-r^2/\varsigma}\tan^{-1}\left(\frac{a\sqrt{r^2-l_1^2}}{l_1\sqrt{a^2-r^2/\varsigma}}\right) + \frac{\pi}{2}\sqrt{a^2-r^2/\bar\varsigma}\right.\right.\\
&\qquad \left.\left. - \frac{z}{\sqrt{\bar\varsigma-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar\varsigma-1}}\right)\right]\right\},\\
\frac{\partial\Psi}{\partial z} &= g_2(z) = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{\sqrt{a^2-r_0^2}}\left[\ln\left(\frac{a+\sqrt{a^2-l_1^2}}{l_1}\right) + \frac{1}{\sqrt{\varsigma-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\varsigma-1}}\right)\right.\\
&\qquad \left. + \frac{1}{\sqrt{\bar\varsigma-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar\varsigma-1}}\right)\right],\\
\frac{\partial^2\Psi}{\partial z^2} &= g_3(z) = \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right],\\
\Lambda^2\Psi &= g_4(z) = -\frac{z(3R_0^2-z^2)}{\bar t^2R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{z}{\bar t}\left[\frac{2}{\bar th} - \frac{ht}{R_0^2(R_0^2+h^2)}\right] - \frac{z\mathrm{e}^{\mathrm{i}\phi}(r^2-r_0^2\varsigma)}{\bar thr(R_0^2+h^2)}\\
&\qquad -\frac{1}{\bar t^2}\left(\frac{\sqrt{a^2-r_0^2}}{\sqrt{a^2-r^2/\varsigma}} + \frac{\sqrt{a^2-r^2/\varsigma}}{\sqrt{a^2-r_0^2}}\right)\tan^{-1}\left(\frac{a\sqrt{r^2-l_1^2}}{l_1\sqrt{a^2-r^2/\varsigma}}\right) + \frac{3z}{\bar t^2\sqrt{a^2-r_0^2}\sqrt{\bar\varsigma-1}}\\
&\qquad \times\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar\varsigma-1}}\right) + \frac{z}{\bar t^2h}\left[\frac{l_1^2(1-1/\bar\varsigma)}{a^2\varsigma-l_1^2} + \frac{a^2-l_1^2}{a^2\bar\varsigma-l_1^2}\right] - \frac{\pi}{\bar t^2\sqrt{a^2-r_0^2}}\sqrt{a^2-r^2/\bar\varsigma},\\
\Lambda\frac{\partial\Psi}{\partial z} &= g_5(z) = \frac{t}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{z^2}{h\bar tR_0^2} - \frac{1}{\sqrt{a^2-r_0^2}\,\bar t\sqrt{\bar\varsigma-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar\varsigma-1}}\right).
\end{aligned} \tag{42}
$$

The first formula in Eq. (42) is obtained by integrating the second equation in Eq. (A.2) in Appendix A. The involved integration is basic but tedious, and some skills such as the change of variable technique should be employed. The detailed derivation is omitted here for simplicity, and the reader is referred to the appendices in Fabrikant (1996) for relative formulations. The fourth formula in Eq. (42) is obtained by direct differentiation of the first formula.

Thus, the expressions of the thermoelastic field of the second part can be obtained as follows:

$$
\begin{aligned}
&U^{(2)} = \frac{2s_3}{\pi}\sum_{i=1}^{3} h_{i2} g_1(z_i)\Theta_0, \quad w^{(2)} = -\frac{2s_3}{\pi}\sum_{i=1}^{3}\alpha_{i1} h_{i2} g_2(z_i)\Theta_0,\\
&\sigma_z^{(2)} = -\frac{2s_3}{\pi}\sum_{i=1}^{3}\gamma_{i1} h_{i2} g_3(z_i)\Theta_0, \quad \sigma_1^{(2)} = -\frac{2s_3}{\pi}\sum_{i=1}^{3}\gamma_{i2} h_{i2} g_3(z_i)\Theta_0,\\
&\sigma_2^{(2)} = \frac{4s_3 c_{66}}{\pi}\sum_{i=1}^{3} h_{i2} g_4(z_i)\Theta_0, \quad \tau_z^{(2)} = -\frac{2s_3}{\pi}\sum_{i=1}^{3}\gamma_{i3} h_{i2} g_5(z_i)\Theta_0,\\
&T^{(2)} = -\frac{2s_3}{\pi}\sum_{i=1}^{3}\alpha_{i2} h_{i2} g_3(z_i)\Theta_0,
\end{aligned}
\tag{43}
$$

where $g_i(z)$ are defined in Eq. (42).

The complete thermoelastic field then can be obtained by superimposing the two parts as given in Eqs. (40) and (43). Now we can discuss the singular behavior at the crack edge of a penny-shaped crack subjected to a point-temperature load arbitrarily acting on the crack surface. Noticing the following property:

$$
z = 0: \quad l_1 \to \min(a, r), \quad \text{and} \quad l_2 \to \max(a, r), \tag{44}
$$

we obtain

$$
z = 0, \ r > a: \quad \sigma_z = -\frac{2s_3 g_{12}}{\pi}\frac{1}{\sqrt{r^2 - a^2}}\frac{\sqrt{a^2 - r_0^2}}{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)}\Theta_0. \tag{45}
$$

If the stress intensity factor is defined as

$$
K_I = \lim_{r\to a}\left\{\sqrt{2\pi(r-a)}\sigma_z|_{z=0}\right\}, \tag{46}
$$

then we have

$$
K_I = -\frac{2s_3 g_{12}}{\sqrt{\pi a}}\Theta_0\frac{\sqrt{a^2 - r_0^2}}{a^2 + r_0^2 - 2ar_0\cos(\phi - \phi_0)}. \tag{47}
$$

At this stage, we can easily obtain the stress intensity factor for a penny-shaped crack subjected to an arbitrarily distributed temperature $\Theta(r, \phi)$ at the crack surface through integration of Eq. (47)

$$
K_I = -\frac{2s_3 g_{12}}{\sqrt{\pi a}}\int_0^{2\pi}\int_0^{a}\frac{\sqrt{a^2 - r_0^2}\,\Theta(r_0, \phi_0)}{a^2 + r_0^2 - 2ar_0\cos(\phi - \phi_0)} r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0. \tag{48}
$$

6. Uniform-temperature load case

Now we consider the particular case of the penny-shaped crack subjected to a uniform temperature T_0 at the crack surface. Tsai (1983a) solved this problem by using Hankel transform and only derived some expressions for the stress and displacement at $z = 0$. In the following, we will derive the complete expressions of the thermoelastic field in the full space. Actually, according to Fabrikant (1989), we can obtain the solutions to Eqs. (30) and (31) as follows:

$$\vartheta(r,\phi) = -\frac{2s_3T_0}{\pi\sqrt{a^2-r^2}}, \quad \omega(r,\phi) = \frac{2s_3g_{12}T_0}{\pi g_{11}}\sqrt{a^2-r^2}. \tag{49}$$

It can be seen that the first equation in Eq. (49) is identical to that obtained by Tsai (1983a). The form of the second equation in Eq. (49) is also the same as that in Tsai (1983a) except for the coefficient. Numerical calculation for certain materials shows that our formula is identical with the one in Tsai (1983a). Substituting Eq. (49) into Eq. (24) yields (Fabrikant, 1989, 1991)

$$\begin{aligned} H_1(r,\phi,z) &= \frac{s_3g_{12}T_0}{g_{11}}\left[(2a^2+2z^2-r^2)\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{2a^2-3l_1^2}{a}\sqrt{l_2^2-a^2}\right],\\ H_2(r,\phi,z) &= -2s_3T_0\left[\left(z^2-a^2-\frac{r^2}{2}\right)\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{3(2a^2-l_1^2)}{2a}\sqrt{l_2^2-a^2} + 2az\ln(l_2+\sqrt{l_2^2-r^2})\right]. \end{aligned} \tag{50}$$

Having obtained H_1 and H_2, the whole thermoelastic field can be obtained simply by differentiation:

$$\begin{aligned} U &= -\frac{2s_3g_{12}T_0}{g_{11}}re^{i\phi}\sum_{i=1}^{3}h_{i1}\left[\frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2} - \sin^{-1}\left(\frac{a}{l_{2i}}\right)\right]\\ &\quad - 4s_3T_0\frac{e^{i\phi}}{r}\sum_{i=1}^{3}h_{i2}\left[\sqrt{l_{2i}^2-a^2}\left(a-\frac{l_{1i}^2}{2a}\right) - z_ia + \frac{r^2}{2}\sin^{-1}\left(\frac{a}{l_{2i}}\right)\right],\\ w &= \frac{4s_3g_{12}T_0}{g_{11}}\sum_{i=1}^{3}\alpha_{i1}h_{i1}\left[z_i\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \sqrt{a^2-l_{1i}^2}\right]\\ &\quad - 4s_3T_0\sum_{i=1}^{3}\alpha_{i1}h_{i2}\left[z_i\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \sqrt{a^2-l_{1i}^2} + a\ln\left(l_{2i}+\sqrt{l_{2i}^2-r^2}\right)\right],\\ \sigma_z &= \frac{4s_3g_{12}T_0}{g_{11}}\sum_{i=1}^{3}\gamma_{i1}h_{i1}\left[\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2-l_{1i}^2}\right] - 4s_3T_0\sum_{i=1}^{3}\gamma_{i1}h_{i2}\sin^{-1}\left(\frac{a}{l_{2i}}\right),\\ \sigma_1 &= \frac{4s_3g_{12}T_0}{g_{11}}\sum_{i=1}^{3}\gamma_{i2}h_{i1}\left[\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2-l_{1i}^2}\right] - 4s_3T_0\sum_{i=1}^{3}\gamma_{i2}h_{i2}\sin^{-1}\left(\frac{a}{l_{2i}}\right),\\ \sigma_2 &= -\frac{8c_{66}s_3g_{12}T_0}{g_{11}}ae^{i2\phi}\sum_{i=1}^{3}h_{i1}\frac{l_{1i}^2\sqrt{l_{2i}^2-a^2}}{l_{2i}^2(l_{2i}^2-l_{1i}^2)} + 4c_{66}s_3T_0\frac{e^{i2\phi}}{r^2}\sum_{i=1}^{3}h_{i2}\left(a\sqrt{l_{2i}^2-a^2} + z_i\sqrt{a^2-l_{1i}^2} - 2z_ia\right),\\ \tau_z &= \frac{4s_3g_{12}T_0}{g_{11}}a^2re^{i\phi}\sum_{i=1}^{3}\gamma_{i3}h_{i1}\frac{\sqrt{a^2-l_{1i}^2}}{l_{2i}^2(l_{2i}^2-l_{1i}^2)} + 4s_3T_0\frac{e^{i\phi}}{r}\sum_{i=1}^{3}\gamma_{i3}h_{i2}\left(\sqrt{a^2-l_{1i}^2} - a\right),\\ T &= \frac{4s_3g_{12}T_0}{g_{11}}\sum_{i=1}^{3}\alpha_{i2}h_{i1}\left[\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2-l_{1i}^2}\right] - 4s_3T_0\sum_{i=1}^{3}\alpha_{i2}h_{i2}\sin^{-1}\left(\frac{a}{l_{2i}}\right), \end{aligned} \tag{51}$$

where

$$l_{1i} = \frac{1}{2}\left[\sqrt{(r+a)^2+z_i^2} - \sqrt{(r-a)^2+z_i^2}\right], \quad l_{2i} = \frac{1}{2}\left[\sqrt{(r+a)^2+z_i^2} + \sqrt{(r-a)^2+z_i^2}\right].$$

Thus we can deduce from Eq. (51)

$$\sigma_z|_{z=0} = -4s_3 g_{12} T_0 a \frac{1}{\sqrt{r^2-a^2}}, \quad (r > a). \tag{52}$$

The above formula is exactly the same as that obtained by Tsai (1983a). Then the stress intensity factor can be obtained as

$$K_I = -4s_3\sqrt{\pi a_0} g_{12} T_0. \tag{53}$$

Note that this formula can also be obtained from Eq. (48) by performing the integration.

7. Conclusions

This paper derives a general solution expressed by four quasi-harmonic functions by the introduction of two displacement functions and using the operator theory. The form of the solution is very simple in that it does not contain a particular part related to the temperature field. It can be used to solve various kinds of mixed boundary-value problems in thermoelasticity, such as the crack problems and punch problems. Although a comprehensive comparison with that of Ashida et al. (1993) is not suitable because of different problems and different ideas adopted, the derivation presented in this paper is somehow simpler and mathematically compact while that in Ashida et al. (1993) needs some skillful techniques and pertinent experience. It is also noted that for isotropic materials, there is a general solution expressed by harmonic functions only (Shail, 1964; Sneddon and Lowengrub, 1969), as mentioned earlier in this paper. However, the temperature field was also first solved by Shail (1964) and the associated harmonic function should be obtained through integration of the temperature.

Using this general solution, the potential theory method proposed by Fabrikant (1989) is extended to solve the problem of a penny-shaped crack subjected to temperature load in an infinite transversely isotropic medium. A new potential is introduced to take the thermal effect into account. Two governing integral or integro-differential equations are derived. It is found that the structure of the integral equation is the same as that for crack problems in elasticity and that of the integro-differential equation is identical to that for contact problems in elasticity. Thus the results obtained by Fabrikant (1989) can be utilized directly. It should be noted that the extension of Fabrikant's theory to thermoelasticity is not straightforward because of the lack of a general solution like the one presented in our paper.

For the problem of a penny-shaped crack subjected to point surface temperature, complete and exact fundamental solution of the thermoelastic field is derived. The corresponding stress intensity factor is derived explicitly, and the one for an arbitrarily distributed temperature load is also presented. These results are new to the literature. The thermoelastic fundamental solution can also play an important role in BEM and defect analysis. In the case of a uniform-temperature load applied on the crack surface, complete and exact expressions of the thermoelastic field are also derived. Most of them have not been obtained before. Good agreement is obtained when compared with some existent formulations obtained by Tsai (1983a), who however, only presented the expressions of stresses and displacements at the plane where the crack is located.

Note that it is very simple and straightforward to give numerical results since we have all the expressions in hand. The present solution method is verified by considering the uniform-temperature load case and comparing some results with that obtained by Tsai (1983a). In addition, the point-temperature load solution is checked by performing the integration involved in Eq. (48) for a uniform-temperature load, which leads to Eq. (53) that is also the same as that obtained by Tsai (1983a). From this view of point, the correctness and validity of our method and the results are verified.

The steady-state problem of a penny-shaped crack subjected to heat flux at the crack surface (may be non-axisymmetric) can also be solved using the present method. However, certain mathematical difficulty will be encountered to perform the integration of new functions that have not appeared before. The work in this respect will be reported in another paper.

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Appendix A

The various derivates of Green's functions $K(r,\phi,z;r_0,\phi_0)$ and $\Pi(r,\phi,z;r_0,\phi_0)$ have been derived by Fabrikant (1989) and are listed in the following for the reader's convenience.

(1) Derivatives of Green's function $K(r,\phi,z;r_0,\phi_0)$

$$\begin{aligned}
&\Lambda K = 2\pi f_1(z) = \frac{2\pi}{\bar{t}}\left[\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{\sqrt{a^2-r_0^2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{\sqrt{l_2^2-a^2}}\right)\right],\\
&\frac{\partial K}{\partial z} = -2\pi f_2(z) = -\frac{2\pi}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right),\\
&\frac{\partial^2 K}{\partial z^2} = 2\pi f_3(z) = 2\pi\left[\frac{z}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{h}{z(R_0^2+h^2)}\left(\frac{r^2-l_1^2}{l_2^2-l_1^2} - \frac{z^2}{R_0^2}\right)\right],\\
&\Lambda^2 K = 2\pi f_4(z) = 2\pi\left\{\frac{\sqrt{a^2-r_0^2}}{\bar{t}\bar{s}}\left(\frac{2}{\bar{t}} - \frac{r_0 e^{i\phi_0}}{\bar{s}^2}\right)\tan^{-1}\left(\frac{\bar{s}}{\sqrt{l_2^2-a^2}}\right) - \frac{z(3R_0^2-z^2)}{\bar{t}^2R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)\right.\\
&\qquad\left. + \frac{\sqrt{a^2-r_0^2}\sqrt{l_2^2-a^2}\,r_0e^{i\phi_0}}{\bar{t}\bar{s}^2[l_2^2 - rr_0e^{-i(\phi-\phi_0)}]} - \frac{zh}{R_0^2+h^2}\left[\frac{t}{\bar{t}R_0^2} - \frac{r^2e^{i2\phi}}{(l_2^2-l_1^2)(l_2^2-r^2)}\right]\right\},\\
&\Lambda\frac{\partial K}{\partial z} = 2\pi f_5(z) = 2\pi\left[\frac{t}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{h}{R_0^2+h^2}\left(\frac{re^{i\phi}}{l_2^2-l_1^2} + \frac{t}{R_0^2}\right)\right],
\end{aligned} \tag{A.1}$$

where

$$t = re^{i\phi} - r_0e^{i\phi_0}, \quad \bar{s} = \sqrt{a^2 - rr_0e^{-i(\phi-\phi_0)}},$$

$$h = \sqrt{a^2-l_1^2}\sqrt{a^2-r_0^2}/a, \quad R_0 = \sqrt{r^2+r_0^2-2rr_0\cos(\phi-\phi_0)+z^2},$$

$$l_1 = \frac{1}{2}\left[\sqrt{(r+a)^2+z^2} - \sqrt{(r-a)^2+z^2}\right], \quad l_2 = \frac{1}{2}\left[\sqrt{(r+a)^2+z^2} + \sqrt{(r-a)^2+z^2}\right].$$

(2) Derivatives of Green's function $\Pi(r,\phi,z;r_0,\phi_0)$

$$
\begin{aligned}
\frac{\partial \Pi}{\partial z} &= \frac{z}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right],\\
\Lambda\Pi &= \frac{t}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)-\frac{z^2}{h\bar{t}R_0^2}-\frac{1}{\sqrt{a^2-r_0^2}t\sqrt{\bar{\varsigma}-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar{\varsigma}-1}}\right),\\
\frac{\partial^2 \Pi}{\partial z^2} &= \frac{1}{R_0^3}\left(1-\frac{3z^2}{R_0^2}\right)\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]+\frac{1}{h(R_0^2+h^2)}\left(\frac{z^2}{R_0^2}-\frac{r^2-l_1^2}{l_2^2-l_1^2}\right),\\
\Lambda\frac{\partial \Pi}{\partial z} &= -\frac{3zt}{R_0^5}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]+\frac{z}{h(R_0^2+h^2)}\left(\frac{r\mathrm{e}^{\mathrm{i}\phi}}{l_2^2-l_1^2}+\frac{t}{R_0^2}\right),\\
\Lambda^2\Pi &= -\frac{3t^2}{R_0^5}\tan^{-1}\left(\frac{h}{R_0}\right)-\frac{t^2h}{R_0^4(R_0^2+h^2)}+\frac{2z^2}{h\bar{t}^2R_0^2}\left(2-\frac{z^2}{R_0^2}\right)\\
&\quad+\frac{1}{\sqrt{a^2-r_0^2}}\left[\frac{3}{\bar{t}^2\sqrt{\bar{\varsigma}-1}}\tan^{-1}\left(\frac{\sqrt{a^2-l_1^2}}{a\sqrt{\bar{\varsigma}-1}}\right)+\frac{a\sqrt{a^2-l_1^2}}{\bar{t}^2(a^2\bar{\varsigma}-l_1^2)}-\frac{a\sqrt{a^2-l_1^2}r^2\mathrm{e}^{\mathrm{i}2\phi}}{l_1^2(R_0^2+h^2)(l_2^2-l_1^2)}\right],
\end{aligned}
\tag{A.2}
$$

where $\varsigma=(r/r_0)\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}$.

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